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THE AGE OF A MARKOV PROCESS*

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The concept of a limiting conditional age distribution of a continuous time Markov process whose state space is the set of non-negative integers and for which {0} is absorbing is defined as the weak limit as $t\to\infty$ of the last time before t an associated "return" Markov process exited from {0} conditional on the state, j, of this process at t. It is shown that this limit exists and is non-defective if the return process is ρ-recurrent and satisfies the strong ratio limit property. As a preliminary to the proof of the main results some general results are established on the representation of the ρ-invariant measure and function of a Markov process. The conditions of the main results are shown to be satisfied by the return process constructed from a Markov branching process and by birth and death processes. Finally, a number of limit theorems for the limiting age as $j\to\infty$ are given.

Markov process	limiting age
ρ-classification	strong ratio limit property
Markov branching process	birth and death process
limit theorems	Green function

1. Introduction

Levikson [20] has recently introduced the concept of the (limiting) conditional age distribution (CAD) of a Markov chain, $\{Z_n\}$ with state space $\mathcal{S}=(0,1,\ldots)$ and a single absorbing state {0}. He does this by defining $T_n=\inf\{m \mid X_{n-m}=0\}$ where the Markov chain $\{X_n\}$, called the return chain, is obtained by concatenating independent realizations of $\{Z_n\}$, that is, transitions of the return chain within, and out of, $\mathcal{T}=\mathcal{S}\backslash\{0\}$ are governed by the same transition probabilities as $\{Z_n\}$, but if $X_n=0$ then $X_{n+1}=1$. Levikson's specification differs slightly from this, but, as we have argued in [24], the present convention seems more convenient. The CAD is the distribution $\{\lim_{n\to\infty} P_i(T_n=k \mid X_n=j)\}$ ($j\in\mathcal{T}$) if this exists; it may be defective ($P_i(A \mid B)=P(A \mid B, A(0)=i)$). Levikson proved that the CAD is well defined and non-defective if the return chain is positive recurrent. The present author [24] has shown that this situation obtains for any return chain which is R-recurrent

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and enjoys the strong ratio limit property (SRLP). He also gave an example showing that the CAD can exist for a R-transient return chain but (may) be defective.

Levikson [20] also introduces certain expressions for return-type diffusions; these are formal analogues of those derived for $\{X_n\}$. He derives some consequences from these and asserts, again by analogy, that they are also valid for discrete state Markov processes.

It is our purpose in this paper to extend the basic results of [24] to Markov processes. As in this reference, we shall initially broaden our scope by considering an appropriate version of a regular Markov process $\{Y(t); t \geq 0\}$ for which $\mathcal{S}$ is irreducible and for a fixed $a \in \mathcal{S}$ define $Y(t, \omega) = t - \sup \overline{S_a(\omega)} \cap [0, t]$ where $\bar{S}_a(\omega)$ is the closure of $S_a(\omega) = \{t \mid Y(t, \omega) = a\}$ and $\omega \in \Omega$, the domain of the random variables $Y(t)(t \geq 0)$; see Chung [5]. We shall then show that provided $\{Y(t)\}$ is ρ-recurrent (Kingman [17]) and satisfies the SRLP, the conditional distribution of $T(t)$ given $\{Y(t) = j\}$ has a non-defective limit. While the proof of this fact is not difficult, it is not the triviality that is the corresponding discrete time proof. We shall then specialize to return processes. This program requires a representation of the ρ-invariant measure and function of the transition matrix $[p_{ij}(t)]$, where $p_{ij}(t) = P_i(Y(t) = j)$, in terms of a Laplace transform of the densities of certain first entrance and last exit times. Such a representation is analogous to the known results for discrete time processes; see [27, p. 165]. This is carried out in Section 2 under either one of two sets of conditions. Firstly we assume that $\{Y(t)\}$ is ρ-recurrent and satisfies the SRLP. For the second set of conditions we replace the SRLP by the requirement that all states of $\{Y(t)\}$ are stable and that it be the minimal process corresponding to its generator.

Equivalent results have recently been obtained by Nummelin [21] who derives a ρ-classification for semi-Markov processes and then applies this to Markov processes. His representations are in terms of a Laplace transform of measures derived from the semi-Markov kernel and their equivalence to those below is not immediately obvious. Our second set of assumptions corresponds with Nummelin's conditions. Both our sets of assumptions are satisfied by the specific cases discussed later in the paper.

In Section 4 we work through the case where $\{Z(t)\}$ is a regular Markov branching process; the CAD can be interpreted as the limiting distribution of the time back to the birth of the first ancestor, given the present population size. In this case the return process is a special case of state dependent immigration. The limit theory and ρ-classification of this class of processes has been examined by Yamazato [33] and Stewart [29], respectively.

In Section 5 we consider regular birth and death processes. In particular we obtain the ρ-classification of these processes by using the spectral representation of the transition probabilities. The criterion given below for ρ-recurrence can also be found in [4, Theorem 1a]. We show that the CAD always exists and is non-defective iff the return process is ρ-recurrent.

It will become apparent that the CAD usually cannot be explicitly specified. One possible solution is to seek limit theorems for the CAD as the present state becomes large. We obtain results of this nature for the Markov branching process in Section 4 and for some particular birth-death processes in Section 6.

Finally we make the obvious point that the results of Section 2, 3 can easily be modified if the minimal state space is a subset of $(0, 1, \ldots)$ and in particular the results hold without change if $\mathcal{S} = (0, 1, \ldots, N)$.

2. Representation theory

As in the introduction we let $\{Y(t); t \geqslant 0\}$ be a Markov process with a countable minimal state space $\mathcal{S}$ and stochastic transition matrices $[p_{ij}(t)]$ for which $p_{ij}(t) > 0$ whenever $t > 0$ and $i, j \in \mathcal{S}$. Then, as is shown by Kingman [17] the limit

$$\rho = -\lim_{t\to\infty} (\log p_{ij}(t))/t$$

exists, is finite and is independent of i and j. We call ρ the decay parameter of $\mathcal{S}$ (and of the process). Furthermore, there exist sequences $\{m_j\}$ and $\{x_j\}$ satisfying the inequalities

$$\sum_{i\in\mathcal{S}} m_i p_{ij}(t) \leqslant e^{-\rho t} m_j \quad (j \in \mathcal{S}), \qquad \sum_{j\in\mathcal{S}} p_{ij}(t) x_j \leqslant e^{-\rho t} x_i \quad (i \in \mathcal{S}). \tag{1}$$

We say that $\{m_j\}$ is a ρ-subinvariant measure and $\{x_j\}$ is a ρ-subinvariant function. Furthermore, we call $\mathcal{S}$ (and the process) ρ-recurrent if $\int_0^\infty e^{\rho t} p_{ii}(t)\,dt = \infty$ for some $i \in \mathcal{S}$ (whence for all $i \in \mathcal{S}$) and ρ-transient otherwise. In the ρ-recurrent case, which will be our primary concern, the ρ-subinvariant measure $\{m_j\}$ and the ρ-subinvariant function $\{x_j\}$ are unique up to constant multiples and are ρ-invariant, that is (1) holds with strict equality. Finally, if $\mathcal{S}$ is ρ-recurrent, we say it (and the process) is ρ-positive if $\lim_{t\to\infty} e^{\rho t} p_{ij}(t) > 0$ for some, and hence for all, $i, j \in \mathcal{S}$. When the limit is zero we say that $\mathcal{S}$ is ρ-null. If $\mathcal{S}$ is 0-recurrent we shall call it recurrent and similarly for the other terms.

In the ρ-positive case $\sum_{i\in\mathcal{S}} m_i x_i < \infty$ and

$$e^{\rho t} p_{ij}(t) \to x_i m_j \Big/ \sum_{i\in\mathcal{S}} m_i x_i \quad (i, j \in \mathcal{S}).$$

In this case

$$\lim_{t\to\infty} p_{ij}(t+\tau)/p_{kl}(t) = e^{-\rho\tau} x_i m_j / x_k m_l \quad (i, j, k, l \in \mathcal{S}; \tau \geqslant 0). \tag{2}$$

Whenever this property holds for some $\rho \geqslant 0$ and sequences $\{m_j\}$ and $\{x_j\}$, we say that the process has the SRLP. If $\mathcal{S}$ is ρ-recurrent then (2) holds iff

$$\limsup_{t\to\infty} p_{00}(t+\tau)/p_{00}(t) \leqslant e^{-\rho\tau} \tag{3}$$

for all sufficiently small positive τ. This follows because $\{Y(t)\}$ has the SRLP if every skeleton $\{Y(nh)\}$, $h>0$ possesses it [19] and (3) implies that Pruitt's [26] necessary and sufficient condition is fulfilled. Observing that the skeleton $\{Y(nh)\}$ is R-recurrent, where $R=\exp(\rho h)$, and that the R-invariant measure and function are $\{m_i\}$ and $\{x_i\}$, respectively, (2) now follows from Pruitt's results via a Croftian theorem [18].

All the results above are analogues of corresponding discrete time results [27]. In this case there are also representation results for the R-invariant measure and function in terms of a generating function of certain taboo probabilities. We shall derive an analogous representation for the present case. Before stating our results we need to establish some notation.

We now assume that $\mathcal{S}=(0,1,\dots)$ and is ρ-recurrent and also that $\{Y(t)\}$ is well-separable and measurable; see [5]. It is possible and convenient to assume that $\{Y(t)\}$ satisfies the condition

$$Y(t)=\liminf_{s\uparrow t} Y(s). \tag{4}$$

Let $a\in\mathcal{S}$. We can always define

$$\gamma_a(t,\omega)=\sup \overline{S_a(\omega)}\cap[0,t],$$

called the last exit time from a before time t. If (4) is satisfied then $Y(\gamma_a(t,\omega))=a$. Let $\Gamma_a(s,t)=P_a(\gamma_a\leq s)$ and $\Gamma_{aj}(s,t)=P_a(\gamma_a\leq s,\ Y(t)=j)$ $(j\neq a)$. Chung [5, Section 11.12] shows that there exist functions $g_{aj}(\cdot)$ $(j\neq a)$ which are measurable, bounded everywhere and satisfy

$$p_{aj}(t)=\int_0^t p_{aa}(t-s)g_{aj}(s)\,\mathrm{d}s \tag{5}$$

and

$$\frac{\partial}{\partial s}\Gamma_{aj}(s,t)=p_{aa}(s)g_{aj}(t-s),\qquad \frac{\partial}{\partial s}\Gamma_a(s,t)=p_{aa}(s)g_a(t-s)$$

where $g_a(s)=\sum_{j\neq a} g_{aj}(s)$, which is continuous and non-increasing in s.

Define

$$\rho_i(\omega)=\inf\{t\mid t>0,\ Y(t,\omega)\neq i\}.$$

and

$$\alpha_{ij}(\omega)=\inf\{t\mid t>\rho_i(\omega),\ T(t,\omega)=j\},$$

called the first entrance time from i to j if $i\neq j$ or the first return time to i if $i=j$. Let $F_{ij}(t)=P_i\{\alpha_{ij}\leq t\}$. Then [5, p. 205] if $i\neq j$, $F_{ij}(\cdot)$ has a continuous derivative $f_{ij}(\cdot)$ and

$$p_{ij}(t)=\int_0^t f_{ij}(s)p_{jj}(t-s)\,\mathrm{d}s\quad (j\neq i). \tag{6}$$

Let $q_{ij}=p'_{ij}(0+)$ and $q_i=-q_{ii}$.

Proposition 1. *Assume that the regular Markov process $\{Y(t)\}$ is ρ-recurrent and satisfies the SLRP* (2). *Then the quantities*

$$d_j = \int_0^\infty e^{\rho s} g_{aj}(s)\,ds \quad (j \neq a), \qquad f_i = \int_0^\infty e^{\rho s} f_{ia}(s)\,ds \quad (i \neq a)$$

are finite. If $d_a = f_a = 1$, then

$$m_j = m_a d_j, \qquad x_i = x_a f_i.$$

The assertions are true if instead of (2) *we assume that all states are stable and that $\{Y(t)\}$ is the minimal process corresponding to $[q_{ij}]$.*

Proof. Assume now that $\{Y(t)\}$ has the SRLP (2). Then (5) and Fatou's lemma implies that

$$\infty > m_j/m_a \geq \int_0^\infty e^{\rho s} g_{aj}(s)\,ds \qquad (j \neq a).$$

Now Kingman [17] has shown that $p_{ij}(t) = O(e^{-\rho t})$ and hence the Laplace transforms

$$\hat{p}_{aj}(\theta) = \int_0^\infty e^{-\theta t} p_{aj}(t)\,dt \qquad (j \in \mathscr{S})$$

exist for $\theta > -\rho$. Moreover from (5) we have

$$\hat{p}_{aj}(\theta)/\hat{p}_{aa}(\theta) = \int_0^\infty e^{-s\theta} g_{aj}(s)\,ds \equiv d_j(\theta)$$

and $d_j(\theta) \uparrow d_j$ $(\theta \downarrow -\rho)$. Using Fubini's theorem we obtain

$$\begin{aligned}\sum_{i \in \mathscr{S}} d_i(\theta) p_{ij}(t) &= (1/\hat{p}_{aa}(\theta)) \int_0^\infty e^{-\theta s} \sum_i p_{ai}(s) p_{ij}(t)\,ds \\ &= \left(e^{\theta t} \int_0^\infty e^{-\theta(t+s)} p_{aj}(t+s)\,ds\right) / \hat{p}_{aa}(\theta) \\ &\leq e^{\theta t} d_j(\theta).\end{aligned}$$

Monotone convergence now yields $\sum_i d_i p_{ij}(t) \leq e^{-\rho t} d_j$, that is, $\{d_j\}$ is ρ-subinvariant. Thus $\{d_j\}$ is ρ-invariant and from our normalization we have $m_j = m_a d_j$ $(j \in \mathscr{S})$.

The proof for the invariant function is similar.

If the states are stable and $\{Y(t)\}$ is the minimal process with the generator $[q_{ij}]$ then the $p_{ij}(t)$ are the unique solutions of the forward and backward systems of differential equations:

$$p'_{ij}(t) = -p_{ij}(t) q_j + \sum_{k \neq j} p_{ik}(t) q_{kj}, \tag{7}$$

$$p'_{ij}(t) = -q_i p_{ij}(t) + \sum_{k \neq i} q_{ik} p_{kj}(t). \tag{8}$$

Define $g_{ii}(t) \overset{d}{=} f_{ii}(t)$. then the system (5) is completed by the equation

$$p_{aa}(t) = e^{-q_a t} + \int_0^t p_{aa}(t-s) g_{aa}(s)\,ds \tag{9}$$

(see [5, Section 11.13] and taking Laplace transforms yields

$$\hat{p}_{aj}(\theta) = \delta_{aj}/(\theta + q_j) + \hat{p}_{aa}(\theta)\hat{g}_{aj}(\theta). \tag{10}$$

It is shown in [17] that $p_{aa}(t) \le e^{-\rho t}$, but irreducibility of $\mathcal{S}$ means that $f_{aa}(t) > 0$ $(t > 0)$ and hence (9) shows that $\rho < q_a$. It follows now from (10) that $\hat{g}_{aa}(-\rho) \le 1$ and $\hat{g}_{aa}(+\rho) = 1$ if $\mathcal{S}$ is ρ-recurrent. Substituting (10) into the Laplace transformed version of (8) and then eliminating $\hat{p}_{aa}(\theta)$ via (10) with $j = a$ eventually leads to the system

$$\hat{g}_{aj}(\theta) = \theta^{-1}(1-\delta_{aj})(1-\hat{g}_{aa}(\theta))q_{aj} + \theta^{-1} \sum_k \hat{g}_{ak}(\theta) q_{kj},$$

which enables us to deduce that $\hat{g}_{aj}(-\rho) < \infty$. Monotone convergence then yields

$$\hat{g}_{aj}(-\rho) = -\rho^{-1} \sum_{k \in \mathcal{S}} \hat{g}_{ak}(-\rho) q_{kj} \quad (j \in \mathcal{S})$$

if $\mathcal{S}$ is ρ-recurrent. Thus $\{d_j\}$ is a ρ-invariant measure for Q and hence for $\{Y(t)\}$ (see Tweedie's [30] proposition 2). The corresponding result for $\{f_i\}$ is proved in the same manner by starting with (6), (10) and (7).

3. The limiting conditional age distribution

We begin this section by assuming that $\{Y(t)\}$ is as in the early part of Section 2, that is, is ρ-recurrent and possesses the SRLP. We shall use the normalization $m_j = d_j$ and $x_j = f_j$.

Theorem 1. *If $\{Y(t)\}$ is ρ-recurrent and has the SLRP then*

$$a_j(t) = \lim_{\tau \to \infty} P_i(T(\tau) \le t \mid Y(\tau) = j)$$

exists, is non-defective and given by

$$a_j(t) = m_j^{-1} \int_0^t e^{\rho s} g_{aj}(s)\,ds.$$

(The corresponding result for discrete time appears as Theorem 1 of [23].)

Proof. Define the taboo probabilities

$$_a p_{ij}(t) = P_i\{Y(s,\omega) \ne a\,(\rho_i < s < t),\ Y(t,\omega) = j\}$$

and hence ${}_ap_{ij}(t) \stackrel{d}{=} 0$ if $j = a$. If $a_{ij}(t, \tau) = P_i(T(\tau) \le t \mid Y(\tau) = j)$, then

$$1 - a_{ij}(t, \tau) = \sum_k p_{ik}(\tau - t)\,{}_ap_{kj}(t)/p_{ij}(\tau) \tag{11}$$

(see [5, p. 200] for the case $i = a$). We wish to let $\tau \to \infty$. Fatou's lemma and the SRLP yield

$$\sum_{k \ne a} (m_k/m_j)\, e^{\rho t}\,{}_ap_{kj}(t) \le \liminf_{\tau \to \infty} (1 - a_{ij}(t, \tau)). \tag{12}$$

We rewrite the right-hand side of (11) using the identity

$${}_ap_{kj}(t) = p_{kj}(t) - \int_0^t p_{aj}(t-u) f_{ka}(u)\, du \quad (k \ne a) \tag{13}$$

and obtain

$$1 - a_{ij}(t, \tau) = 1 - p_{ia}(\tau - t) p_{aj}(t)/p_{ij}(\tau)$$

$$- \sum_{k \ne a} (p_{ik}(\tau - t)/p_{ij}(\tau)) \int_0^t p_{aj}(t-u) f_{ka}(u)\, du.$$

Using Fatou's lemma and the SRLP again yields

$$\limsup_{\tau \to \infty} (1 - a_{ij}(t, \tau)) \le 1 - e^{\rho t} p_{aj}(t)/m_j$$

$$- \int_0^t e^{\rho t} p_{aj}(t-u) \sum_{k \ne a} (m_k/m_j) f_{ka}(u)\, du. \tag{14}$$

However, using (13) again we obtain

$$e^{-\rho t} m_j - p_{aj}(t) = \sum_{k \ne a} m_k p_{kj}(t)$$

$$= \sum_{k \ne a} m_{ka} p_{kj}(t) + \int_0^t p_{aj}(t-u) \sum_{k \ne a} m_k f_{ka}(u)\, du$$

and hence the left-hand side of (12) and the right-hand side of (14) are equal. This proves the existence of

$$a_j(t) = \lim_{\tau \to \infty} a_{ij}(t, \tau)$$

$$= 1 - \sum_{k \ne a} (m_k/m_j)\, e^{\rho t}\,{}_ap_{kj}(t).$$

We now proceed to show that $a_j(t)$ is non-defective by obtaining a more convenient representation. We have proved strict equality in (14). The ρ-recurrence shows that the first term on the right has a Laplace transform for positive values of the argument and hence Fubini's theorem gives

$$\hat{a}_j(\theta) = m_j^{-1}\hat{p}_{aj}(\theta-\rho)(1+\sum_{k\neq a} m_k\hat{f}_{ka}(\theta-\rho)).$$

Equation (6) yields $\hat{f}_{ka}(\theta) = \hat{p}_{ka}(\theta)/\hat{p}_{aa}(\theta)$ and hence the Laplace transformed equations for the ρ-invariant measure yield

$$1+\sum_{k\neq a} m_k\hat{f}_{ka}(\theta-\rho) = 1/\hat{p}_{aa}(\theta-\rho).$$

This, together with the Laplace transformed version of (5), yields the asserted form of $a_j(\cdot)$. Clearly it is non-defective and absolutely continuous.

Consider now $\{Z(t), t\geq 0\}$, a regular minimal Markov process with state space $\mathcal{S} = \{0, 1, \ldots\}$ for which $\{0\}$ is the only absorbing state and is accessible from the irreducible set $\mathcal{T} = \mathcal{S}\backslash\{0\}$. Let $[r_{ij}(t)]$ denote the transition semi-group of $\{Z(t)\}$ and $U = [u_{ij}]$ its generator. We shall now define the (elementary) return process $\{X(t)\}$ which will shortly play the role of the $\{Y(t)\}$ process. Let $q_0 \in (0, \infty)$ be given. Define a generator Q by $q_{ij} = u_{ij} (i \in \mathcal{T})$, $q_{00} = -q_0$, $q_{01} = q_0$ and $q_{0j} = 0$ $(j = 2, 3, \ldots)$ and let $\{X(t)\}$ be the minimal process, necessarily unique, constructed from Q. Equivalently, if $X(t) > 0$ the return process evolves according to the construction of $\{Z(t)\}$ until it next hits $\{0\}$. It sojourns there for a time which has an exponential distribution with mean q_0^{-1} and then jumps to $\{1\}$ and evolves as before. We now prove

Theorem 2. *If the return process is ρ-recurrent and has the SRLP then the CAD, $a_j(\cdot)$, exists, is non-defective and given by*

$$a_j(t) = \int_0^t e^{\rho u} r_{1j}(u)\,du \Big/ \int_0^\infty e^{\rho u} r_{1j}(u)\,du.$$

Its Laplace-Stieltjes transform $\alpha_j(\theta) = \int_0^\infty e^{-\theta t} a_j(dt)$ is given by

$$\alpha_j(\theta) = \hat{p}_{0j}(\theta-\rho)/\hat{p}_{00}(\theta-\rho) = \hat{r}_{1j}(\theta-\rho)/\hat{r}_{1j}(\rho).$$

Remark. Levikson [20] has asserted, without proof, these results when $\rho = 0$.

Proof. Let the notation introduced for $\{Y(t)\}$ now apply to the return process but with $a = 0$. Clearly

$$F_{00}(t) = q_0 \int_0^t e^{-q_0 u} r_{10}(t-u)\,du \tag{15}$$

and hence solving (9) shows that

$$p_{00}(t) = \int_0^t e^{-q_0(t-u)}\,d\mathcal{F}_{00}(u) \tag{16}$$

where $\mathcal{F}_{00}(\cdot)$ is the renewal function generated by $F_{00}(\cdot)$. Furthermore

$${}_0p_{0j}(t) = q_0 \int_0^t e^{-q_0 u} r_{1j}(t-u)\,du \quad (j\neq 0)$$

and hence from (16) and the renewal equation

$$p_{0j}(t) = {}_0p_{0j}(t) + \int_0^t p_{0j}(t-u) f_{00}(u)\,\mathrm{d}u$$

[5, p. 189] we obtain

$$\hat{p}_{00}(\theta) = \hat{f}_{00}(\theta)/(q_0+\theta),$$

$$\hat{p}_{0j}(\theta) = q_0 \hat{r}_{1j}(\theta) \hat{f}_{00}(\theta)/(q_0+\theta) \quad (j \neq 0).$$

Since, however,

$$\hat{g}_{0j}(\theta) = \hat{p}_{0j}(\theta)/\hat{p}_{00}(\theta) \quad (j \neq 0)$$

we finally obtain

$$g_{0j}(t) = q_0 r_{1j}(t) \quad (j \neq 0),$$

and Theorem 2 follows.

We now derive expressions for the moments of the CAD. These generalise an assertion of Levikson's [20] for the first order moment when $\{X(t)\}$ is a recurrent birth-death process.

Let

$$\mu_j^{(n)} = \int_0^\infty t^n a_j(\mathrm{d}t) = (G_{1j}(\rho))^{-1} \int_0^\infty t^n \,\mathrm{e}^{\rho t} r_{1j}(t)\,\mathrm{d}t$$

where

$$G_{ij}(\rho) = \int_0^\infty \mathrm{e}^{\rho t} r_{ij}(t)\,\mathrm{d}t \quad (i, j \in \mathscr{T}).$$

The $G_{ij}(\rho)$ are finite. This is seen by observing that $r_{ij}(t) = {}_0p_{ij}(t)$ and considering the process with transition probabilities $\tilde{p}_{ij}(t) = \mathrm{e}^{\rho t} p_{ij} x_j / x_i$. Clearly ${}_0\tilde{p}_{ij}(t) = \mathrm{e}^{\rho t}{}_0p_{ij}(t) x_j/x_i$ and since $\{0\}$ is accessible from $\{j\}$, $\int_0^\infty {}_0\tilde{p}_{ij}(t)\,\mathrm{d}t < \infty$, see [5, p. 192]. Let now

$$G_{ij}^{(n)}(\rho) = \sum_{k_1,\ldots,k_{n-1} \geqslant 1} G_{ik_1}(\rho) G_{k_1k_2}(\rho) \cdots G_{k_{n-1}j}(\rho) \quad (n \geqslant 2).$$

Fubini's theorem and the Chapman-Kolmogorov equation yield

$$\begin{aligned} G_{1j}^{(n)}(\rho) &= \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{\rho(t_1+\cdots+t_n)} r_{1j}(t_1+\cdots+t_n)\,\mathrm{d}t_1 \cdots \mathrm{d}t_n \\ &= \int_0^\infty \cdots \int_0^\infty \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \int_{t_1+\cdots+t_{n-1}}^\infty \mathrm{e}^{\rho y} r_{1j}(y)\,\mathrm{d}y \\ &= \int_0^\infty \mathrm{e}^{\rho y} r_{1j}(y)\,\mathrm{d}y \int_{\{t_1+\cdots+t_{n-1} \leqslant y\}} \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}. \end{aligned}$$

It is easily checked that the inner integral equals $y^{n-1}/(n-1)!$ whence

$$\mu_j^{(n)} = n!G_{1j}^{(n+1)}(\rho)/G_{ij}(\rho).$$

The hypotheses of Theorem 2 *are* satisfied if the return process is positive recurrent; the limiting-stationary distribution is given by

$$\pi_0 = \left[1 + \int_0^\infty (1 - r_{10}(t))\,\mathrm{d}t\right]^{-1}, \qquad \pi_j = \pi_0 G_{1j}(0) \quad (j \in \mathcal{T}).$$

4. Markov branching processes

In this section we let $\{Z(t)\}$ be the Markov branching process [3, Chap. 3] whose generator is given by $u_{ij} = \nu i p_{j-i+1}$ $(i \neq j)$ and $u_{ii} = -\sum_{k \neq i} u_{ik}$ where $\nu > 0$, $p_j \geq 0$, $\sum_{j \geq 0} p_j = 1$, $0 < p_0 < 1$ and $p_1 = 0$. As is well known $Z(t)$ represents the size of a population of individuals whose lifetimes are independent and exponentially distributed with mean ν^{-1}, and at the end of its lifetime an individual produces j progeny with probability p_j. All individuals reproduce independently. Clearly $\mathcal{T}$ is irreducible and $\{0\}$ is accessible from $\mathcal{T}$ since $p_0 > 0$. We shall use the notation $G_{ij} = G_{ij}(0)$ and $f(s) = \sum_{i \in \mathcal{S}} p_i s^i$. Regularity of the Markov branching process is equivalent to the condition $\int_{1-\varepsilon}^1 \mathrm{d}s/(f(s) - s) = \infty$ for each ε in $(0, 1-q)$ where q is the probability of eventual absorption when $Z(0) = 1$ and is the least positive root of the equation $f(s) = s$. Let $m = f'(1-)$.

We now determine when the hypotheses of Theorem 2 are fulfilled.

Theorem 3. *The hypotheses of Theorem* 2 *are fulfilled if*

(i) $m < 1$; *or*

(ii) $m = 1$ *and* (a) $\int_0^1 (1-s)/(f(s)-s)\,\mathrm{d}s < \infty$, *or* (b) $f(s) = (1-s)^2 L((1-s)^{-1})$ *where* $L(\cdot)$ *is slowly varying* (*SV*) *at infinity and* $\int_1^\infty \mathrm{d}x/xL(x) = \infty$; *or*

(iii) $m > 1$.

Proof. (i) In this case the expected time to extinction is finite and hence the return process is positive recurrent.

(ii) The return process is recurrent and the expected entrance time, T_1, to $\{0\}$ from $\{1\}$ is finite iff (a) holds. In this case the return process is positive recurrent.

If (b) holds then $E\,T_1 = \infty$; this condition is satisfied in the important special case where $\gamma = \frac{1}{2} f''(1-) < \infty$. As is well known [3] the backward equation for $\{Z(t)\}$ can be integrated and in particular

$$\int_0^{r_{10}(t)} \mathrm{d}u/a(u) = t \tag{17}$$

where $a(s) = \nu(f(s) - s)$ and if $V(t) = \int_1^t dx/L(x)$, (17) becomes

$$V((1 - r_{10}(t))^{-1}) = \nu t.$$

Now $V(\cdot)$ is strictly increasing and $\sim t/L(t)$ $(r \to \infty)$ and hence we find that

$$1 - r_{10}(t) = 1/\nu t M(\nu t) \tag{18}$$

where $M(\cdot)$ is SV at infinity and satisfies $M(t)/L(tM(t)) \to 1$, $L(t)/M(t/L(t)) \to 1$ $(t \to \infty)$. We now show that

$$1 - F_{00}(t) \sim 1 - r_{10}(t) \quad (t \to \infty). \tag{19}$$

This follows by observing first that $F_{00}(t)$ is the convolution of the DF's $1 - e^{-q_0 t}$ and $r_{10}(t)$ whence the inequality

$$\begin{aligned}(1 - r_{10}(t') + e^{-q_0 t'})(1 - \varepsilon) &\leq 1 - F_{00}(t) \\ &\leq (1 - r_{10}(t'') + e^{-q_0 t''})(1 + \varepsilon)\end{aligned}$$

where $t' = (1 - \delta)t$, $t'' = (1 + \delta)t$, $0 < \delta, \varepsilon < 1$ and the inequality holds for all t sufficiently large; see [9, p. 278]. Since $e^{-q_0 t}/(1 - r_{10}(t)) \to 0$ $(t \to \infty)$ (19) readily follows. Let $m(t) = \int_0^t (1 - F_{00}(u))\, du$. It follows from (18) and (19) that $m(\cdot)$ is SV at infinity and hence we can apply a renewal theorem of Erickson [7] to (16) and obtain

$$p_{00}(t) \sim 1/q_0 m(t) \tag{20}$$

Thus (3) is satisfied with $\rho = 0$ and hence the conditions of Theorem 2 are again fulfilled. When $\gamma < \infty$, $L(x) \to \gamma$ $(x \to \infty)$ and (20) becomes

$$p_{00}(t) \sim \nu\gamma/q_0 \log t$$

which is implicitly contained in [33].

(iii) Consider now the case where $m > 1$. Stewart [29] has shown that the function

$$g(\theta) = \theta\left[1 + q_0 \int_0^\infty e^{-\theta t}(1 - r_{10}(t))\, dt\right]$$

exists and is strictly increasing for $\theta > d = \max(-q_0(1 - q), a'(q))$, $d < 0$, and there exists $\rho \varepsilon (0, -d)$ such that $g(-\rho) = 0$. He then shows that $\{X(t)\}$ is ρ-positive and hence Theorem 2 applies here also.

The following result contains some information on the moments of the CAD of the Markov branching process.

Proposition 2. (a) *If $m \neq 1$ the CAD has moments of all orders.*

(b) *When $m = 1$ then $\mu_i^{(1)} < \infty$ iff $ET_1 < \infty$.*

(c) *If $m<1$, or $m=1$ and $ET_1<\infty$, then $\mu_i^{(1)}$ is determined through*

$$\sum_{j=1}^{\infty} G_{1j}s^j=\int_0^s \frac{(1-x)}{a(x)}\,\mathrm{d}x$$

and

$$\sum_{j=1}^{\infty} G_{1j}^{(2)}s^j=\int_0^s \frac{1}{a(x)}\int_x^1 \frac{(1-y)}{a(y)}\,\mathrm{d}y\,\mathrm{d}x.$$

Proof. (a) this follows since $r_{1j}(t)$ decays exponentially fast; [3, p. 115] and [34].

(b) This follows from (19).

(c) Let

$$G_i(s)=\sum_{j=1}^{\infty} G_{ij}s^j, \qquad F_i(s,t)=\sum_{j=0}^{\infty} r_{ij}(t), \qquad F_i'=(\partial/\partial s)F_i.$$

Using Fubini's theorem and the forward equations for $\{Z(t)\}$ we obtain

$$\begin{aligned} G_i(s)&=\int_0^\infty (F_i(s,t)-F_i(0,t))\,\mathrm{d}t\\ &=\int_0^\infty\int_0^s F_i'(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^\infty\int_0^s \frac{(\partial/\partial t)F_i(x,t)}{a(x)}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^s \frac{q^i-x^i}{a(x)}\,\mathrm{d}x<\infty \quad (0\le s<1). \end{aligned} \tag{21}$$

Now let $m\le 1$ and $G^{(2)}(s)=\sum_{j=1}^{\infty} G_{1j}^{(2)}s^j$ and observe that $G=G_1(1-)<\infty$ if either $m<1$ or $m=1$ and $E(T)<\infty$ since $G=ET$. For these cases use of Fubini's theorem and (21) yield the relation

$$\begin{aligned} G^{(2)}(s)&=\int_0^s \frac{(G-G_1(x))}{a(x)}\,\mathrm{d}x\\ &=\int_0^s\left(\frac{1}{a(x)}\right)\int_x^1 \frac{1-y}{a(y)}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Let $A(j)$ be the limiting conditional age, that is, it is a random variable whose DF is the CAD. We shall now obtain some limit theorems for $A(j)$ for large j.

We shall first consider the case $m=1$ and prove the following preliminary result which is of some interest in itself.

Proposition 3. *If $f(s)-s=(1-s)^{1+\eta}L((1-s)^{-1})$ where $0<\eta\le 1$ and $L(\cdot)$ is SV at infinity, then*

$$G_{ij}\sim i/\nu\Gamma(\eta)j^{2-\eta}L(j) \quad (j\to\infty).$$

Proof. It follows from (21) that

$$G_i'(s) = \frac{1-s^i}{(1-s)(1-W(s))}$$

where $W(s) = (1-f(s))/(1-s)$ is a probability generating function and hence we obtain the representation

$$\nu j G_{ij} = v_{j+1} + \cdots + v_{j-i}$$

where

$$\sum_{j=0}^{\infty} v_j s^j = \frac{1}{(1-W(s))}.$$

Our assertion now follows from renewal theorems for the v_j; see [25] for the details.

Theorem 4. *If $m = 1$, $\sum_{j\geq 1} p_j j^2 \log j < \infty$ and $\sigma = \nu f''(1-)/2$, then*

$$\lim_{j\to\infty} P(A(j) \leq jt) = \exp(-1/\sigma t) \quad (t > 0).$$

Proof. The hypotheses allow us to invoke a local limit theorem [6] which states that if $\sigma = \nu\gamma$ then

$$\sigma^2 t^2 (\exp(j/\sigma t)) r_{ij}(t) \to 1$$

as $t, j \to \infty$ but such that j/t is bounded. Using this result with Proposition 3 shows that if $j/t = 0(1)(j \to \infty)$ then $G_{1j}^{-1} r_{1j}(t)$ can be replaced by $j/\sigma t^2 \exp(-j\sigma t)$ with negligible error as $j \to \infty$ and hence

$$G_{1j}^{-1} \int_{jt}^{\infty} r_{1j}(s)\,\mathrm{d}s \sim (j/\sigma) \int_{jt}^{\infty} \mathrm{e}^{-j/\sigma s}\,\mathrm{d}s/s^2 = 1 - \mathrm{e}^{-1/\sigma t}.$$

For the case $1 < m < \infty$ we need to review some well-known facts. There is a function $c(t)$ $(0 < t < \infty)$ satisfying $c(t) \uparrow \infty$, $c(t+\tau)/c(t) \to \mathrm{e}^{\Delta\tau}$, $\Delta = \nu(m-1)$, $(t \to \infty, \tau \geq 0)$ and if $Z(0) = 1$, $Z(t)/c(t) \xrightarrow{\text{a.s.}} W$, a non-defective random variable whose DF has an atom of size q at the origin and is absolutely continuous with a positive density function, $w(\cdot)$, on the set of positive numbers. If $\sum_{j\geq 1} p_j j \log j < \infty$ we may choose $c(t) = \mathrm{e}^{\Delta t}$; see [3, 22, 34]. With this notation we shall prove the following result.

Theorem 5. *Let $1 < m < \infty$, $f''(1-) < \infty$ and $-\infty < x_1 < x_2 < \infty$. Then*

$$P(x_1 < \Delta A(j) - \log j \leq x_2) \to M^{-1} \int_{\exp(-x_2)}^{\exp(-x_1)} t^{-\rho/\Delta} w(t)\,\mathrm{d}t \quad (j \to \infty)$$

where $M = \int_0^{\infty} t^{-\rho/\Delta} w(t)\,\mathrm{d}t.$

This theorem is analogous to the discrete time version in [24] except that in this reference $j \to \infty$ through a lacunary subset of the positive integers. The main component of the proof is the determination of the behavior of $G_{1j}(\rho)$ when j is large. In the following proposition ρ can be any number such that $0 \leq \rho < \min(\delta, \Delta)$ although in our use of the proposition ρ will be the convergence parameter, which $<\delta$.

Proposition 4. *Let* $1 < m < \infty$. *Then (i) if* $\{p_j\}$ *is aperiodic*

$$jG_{1j} \to (1-q)/\Delta$$

and (ii) if $0 < \rho < \min(\delta, \Delta)$ *and* $\sum_{j \geq 1} p_j j \log j < \infty$, *then*

$$j^{1-\rho/\Delta} G_{1j}(\rho) \to \Delta^{-1} \int_0^\infty t^{-\rho/\Delta} w(t)\, \mathrm{d}t < \infty.$$

Proof. (i) Equation (21) yields

$$G_1'(s) = (q-s)/a(s) = 1/\nu(1-w(s))$$

where $w(s) = (f(s)-q)/(s-q)$ is a probability generating function and $w'(1-) = (m-1)/(1-q)$. The assertion now follows from the discrete renewal theorem [8].

(ii) We will derive two representations for $G(\rho, s) = \sum_{j=1}^\infty G_{1j}(\rho)s^j$. The first will give us the asymptotic behavior of $G(\rho, s)$ as $s \uparrow 1$ and the second will allow us to infer the desired result. First observe [3] that $(F_1(s, t) - q) \sim h(s)\exp(-\delta t)$ $(t \to \infty)$ where $h(\cdot)$ is finite for $0 \leq s < 1$. Application of Fubini's theorem then shows that $G(\rho, s) < \infty$ if $0 \leq s < 1$. Let $q \leq s < 1$. Observing that

$$G(\rho, s) = \int_0^\infty e^{\rho t}(F_1(s, t) - q)\, \mathrm{d}t + \int_0^\infty e^{\rho t}(q - F_1(0, t))\, \mathrm{d}t$$

makes it clear that it suffices to consider the first term on the right, $H(\rho, s)$, in order to obtain the asymptotic behavior of $G(\rho, \cdot)$.

Proceeding as for (21) we obtain

$$\begin{aligned} H(\rho, s) &= \int_q^s \mathrm{d}x/a(x) \int_0^\infty e^{\rho t}(\partial/\partial t)F_1(x, t)\, \mathrm{d}t \\ &= \int_q^s ((q-x)/a(x))\, \mathrm{d}x - \rho \int_q^s (H(\rho, x)/a(x))\, \mathrm{d}x \end{aligned} \tag{22}$$

where the second equality follows from integration by parts of the inner integral which can be written as $\int_0^\infty e^{\rho t}\, \mathrm{d}_t(F_1(x, t) - q)$ Differentiation of (22) yields

$$(\partial/\partial s)H(\rho, s) = (q-s)/a(s) - \rho H(\rho, s)/a(s). \tag{23}$$

Choose $s_0 \in (q, 1)$ and let $V(s) = \exp \rho \int_{s_0}^s \mathrm{d}x/a(x)$. Observe now that for $s - q$ small enough,

$$a(s) = -1/\delta(q-s) + \mathrm{O}(1).$$

It follows that

$$V(s) \sim K(s-q)^{-\rho/\delta} \quad (s \downarrow q) \tag{24}$$

where $0<K<\infty$ and hence, since $\rho<\delta$, $V(\cdot)$ is integrable over $[q, s], q<s<1$. Equation (23) can now be solved in the standard way:

$$H(\rho, s)V(s) = -\int_q^s ((x-q)/a(x))V(x)\,\mathrm{d}x. \tag{25}$$

Now use the fact that $V(x)/a(x) = V'(x)/\rho$ and (24) to integrate the right hand side of (25):

$$H(\rho, s)V(s) = -(s-q)V(s)/\rho + \rho^{-1}\int_q^s V(x)\,\mathrm{d}x.$$

Letting $f^*(s) = (m-1)/(f(s)-s) + (1-s)^{-1}$ we have

$$1/a(s) = -1/\Delta(1-s) + \Delta^{-1}f^*(s)$$

and moreover

$$\int_s^1 f^*(s)\,\mathrm{d}s < \infty \quad (q<s<1) \quad \text{iff} \quad \sum_{j\geq 1} p_j j \log j < \infty$$

[3, p. 117]. Thus we obtain

$$V(s) = \left(\frac{1-s}{1-s_0}\right)^{\rho/\Delta} \exp\left((\rho/\Delta)\int_{s_0}^s f^*(x)\,\mathrm{d}x\right).$$

Let $\phi(\theta) = E(\mathrm{e}^{-\theta W})$. Using equations (7) and (11) in [3, pp. 116–117], we obtain the representation

$$H(\rho, s) = -\rho^{-1}(s-q) + \rho^{-1}\int_q^s (\phi^{-1}(x)/\phi^{-1}(s))^{\rho/\Delta}\,\mathrm{d}x$$

and, as is shown in [3, p. 116],

$$\phi^{-1}(s) = (1-s)\exp\left(-\int_s^1 f^*(x)\,\mathrm{d}x\right).$$

It is now apparent that

$$G(\rho, s) \sim \rho^{-1}(1-s)^{-\rho/\Delta}\int_q^1 (\phi^{-1}(x))^{\rho/\Delta}\,\mathrm{d}x$$

and the integral is finite. Making the change of variables $x = \phi(\theta)$ in the integral shows that it equals

$$-\int_0^\infty \theta^{\rho/\Delta}\phi'(\theta)\,\mathrm{d}\theta = \int_0^\infty \theta^{\rho/\Delta}\,\mathrm{d}\theta \int_0^\infty t\,\mathrm{e}^{-\theta t}w(t)\,\mathrm{d}t$$

$$= \Gamma(1+\rho/\Delta)\int_0^\infty t^{-\rho/\Delta}w(t)\,\mathrm{d}t.$$

When $0 \leq s < q$ the procedure leading to (23) yields

$$(\partial/\partial s)G(\rho, s) = (q-s)/a(s) + \rho G(\rho, q)/a(s) - \rho G(\rho, s)/a(s),$$

that is

$$\nu(\partial/\partial s)G(\rho, s) = (1 + \rho G(\rho, q)/(q-s) - \rho G(\rho, s)/(q-s))/(1-w(s)). \quad (26)$$

Let $\sum_{j=0}^{\infty} v_j s^j = 1/(1-w(s))$ so that $\{v_j/v_0\}$ is a renewal sequence; $v_0 = 1 - p_0/q > 0$ and we know that $v_j \to v = (1-q)/(m-1)$ $(j \to \infty)$. Writing

$$g_j = \sum_{k \in \mathscr{S}}^{\infty} G_{1,j+k}(\rho) q^k$$

it is apparent that

$$(G(\rho, q) - G(\rho, s))/(q-s) = q^{-1} \sum_{j \in \mathscr{S}}^{\infty} g_j s^j \quad (0 \leq s < 1)$$

and hence it follows from a Tauberian theorem for power series [9, p. 447] and the results above that

$$\sum_{k=0}^{j} g_k \sim (q/\nu(1-q)) \left(\int_0^{\infty} t^{-\rho/\Delta} w(t)\, dt \right) j^{\rho/\Delta}.$$

However

$$\nu j G_{1j}(\rho) = v_j + (\rho/q) \sum_{k=0}^{j} v_k g_{j-k}$$

and it is easily shown that

$$\left(\sum_{k=0}^{j} v_k g_{j-k} \right) \Big/ \left(\sum_{k=0}^{j} g_k \right) \to v,$$

see [11, p. 42]. This completes the proof.

To complete the proof of Theorem 5 we use a local limit theorem [2] which states that when $f''(1-) < \infty$ and $j, t \to \infty$ in such a way that $0 < c_1 \leq j\, e^{-\Delta t} \leq c_2 < \infty$ then

$$r_{1j}(t) = e^{-\Delta t}[w(j\, e^{-\Delta t}) + O(1)].$$

This shows that

$$\int_{\Delta^{-1}(x_1 + \log j)}^{\Delta^{-1}(x_2 + \log j)} e^{\rho s} r_{1j}(s)\, ds$$
$$\sim \int_{\Delta^{-1}(x_1 + \log j)}^{\Delta^{-1}(x_2 + \log j)} e^{(\rho - \Delta)s} w(j\, e^{-\Delta s})\, ds$$
$$= \Delta^{-1} j^{-(1-\rho/\Delta)} \int_{\exp(-x_2)}^{\exp(-x_1)} t^{-\rho/\Delta} w(t)\, dt$$

and the theorem follows immediately.

We consider now the case $m<1$ and assume that the radius of convergence of $\Sigma p_j s^j$ is sufficiently large for the existence of $\sigma>1$ such that $f(\sigma)=\sigma$. It is easily seen that $\tilde{F}(s,t)=F_1(s\sigma,t)/\sigma$ is the probability generating function of $\{\tilde{r}_{1j}(t)\}$ which defines a super-critical Markov branching process with lifetime mean ν^{-1}, offspring mean $f'(\sigma-)\leq\infty$ and extinction probability $q=\sigma^{-1}$. We thus have

$$G_{1j}=\tilde{G}_{1j}q^{j-1},\qquad r_{1j}(t)=\tilde{r}_{1j}(t)q^{j-1}.$$

Assuming that $f'(\sigma-)<\infty$. Proposition 4 shows that

$$j\sigma^j G_{1j}\to(\alpha-1)/\nu(f'(\sigma-)-1)$$

and the next theorem follows easily:

Theorem 6. *If $m<1$, $f''(\sigma-)<\infty$, $-\infty<x_1<x_2<\infty$ and $D=\nu(f'(\sigma)-1)$, then*

$$\lim_{j\to\infty}P(x_1<DA(j)-\log j<x_2)=(1-\sigma^{-1})^{-1}\int_{\exp(-x_2)}^{\exp(-x_1)}\tilde{w}(t)\,dt$$

where

$$\int_0^\infty e^{-\theta t}\tilde{w}(t)\,dt+\alpha-1=\lim_{t\to\infty}\tilde{F}(\exp(-\theta\,e^{-Dt}),t).$$

In Section 6 we shall present, as a special case, the linear birth and death process.

5. Birth and death processes

In this section we study the birth and death process $\{X(t)\}$ with birth and death parameters $0<\lambda_j<\infty$ $(j\in\mathscr{S})$ and $0<\mu_j<\infty$ $(j\in\mathscr{T})$, respectively. Let $\pi_0=1$ and $\pi_j=(\lambda_0\lambda_1\cdots\lambda_{j-1})/(\mu_1\mu_2\cdots\mu_j)$. It is known [12, 13] that if

$$\sum_{j=0}^{\infty}(\pi_j+1/\lambda_j\pi_j)=\infty \tag{27}$$

there is exactly one standard substochastic transition semi-group whose generator is given by

$$\begin{aligned}
&q_{jj}=-(\lambda_j+\mu_j) && (\mu_0=0)\\
&q_{j,j+1}=\lambda_j && (j\in\mathscr{S})\\
&q_{j,j-1}=\mu_j && (j\in\mathscr{T})\\
&q_{ij}=0, && |i-j|>1
\end{aligned}$$

and which satisfies the forward and backward systems of differential equations and also the Chapman-Kolmogorov equation. If also

$$\sum_{j=0}^{\infty}(\lambda_j\pi_j)^{-1}\sum_{i=0}^{j}\pi_i=\infty \tag{28}$$

then $[p_{ij}(t)]$ is stochastic. We shall always assume that (27) and (28) are in force.

Clearly $\{X(t)\}$ is a return process obtained from the process $\{Z(t)\}$ with the generator defined as above except that $\lambda_0 = 0$, which renders $\{0\}$ absorbing for $\{Z(t)\}$. As before let $\{r_{ij}(t)\}$ denote the transition semi-group of $\{Z(t)\}$; it is stochastic.

We shall show that the CAD always exists but is non-defective iff $\{X(t)\}$ is ρ-recurrent. But first we must discuss the ρ-classification of $\{X(t)\}$ and this is most conveniently done in terms of the spectral representation of its transition semi-group.

Associated with Q is the system of polynomials $\{Q_j(x)\}$ defined by

$$Q_{-1}(x) \stackrel{d}{=} 0, \qquad Q_0(x) \stackrel{d}{=} 1,$$
$$-xQ_j(x) = \mu_j Q_{j-1}(x) - (\lambda_j + \mu_j)Q_j(x) + \lambda_j Q_{j+1}(x) \quad (j \in \mathscr{S}); \tag{29}$$

$Q_j(\cdot)$ is of exact degree j.

There is exactly one probability measure ψ supported in $[0, \infty)$ such that

$$\pi_j \int_0^\infty Q_i(x)Q_j(x)\psi(\mathrm{d}x) = \delta_{ij} \quad (i, j \in \mathscr{S}), \tag{30}$$

that is, $\{Q_j(x)\}$ is orthogonal with respect to ψ and we say that ψ is a (in our case unique) solution to the Q-moment problem. Moreover we have the representation

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x)Q_j(x)\psi(\mathrm{d}x)$$

and hence we call ψ the spectral measure of the process. We shall need the following properties of the $Q_j(\cdot)$: $Q_j(0) = 1$ $(j \in \mathscr{S})$ and if $x < 0$, $1 = Q_0(x) < Q_1(x) < \cdots < Q_j(x) \uparrow \infty$ $(j \to \infty)$. See [12] for details.

Let ρ be the smallest point in the support of ψ; $0 \leq \rho < \infty$ and in addition let ϕ be the measure defined by $\phi(E) = \psi(E + \rho)$, $E \subset [0, \infty)$. Clearly

$$p_{00}(t) = e^{-\rho t} \int_0^\infty e^{-tx} \phi(\mathrm{d}x)$$

and hence for any $\varepsilon > 0$ we have the inequality

$$0 < e^{-(\rho+\varepsilon)t} \phi([0, \varepsilon]) \leq p_{00}(t) \leq e^{-\rho t}.$$

It follows that $-t^{-1} \log p_{00}(t) \to \rho$ $(t \to \infty)$ and hence ρ is the convergence parameter of $\mathscr{S}$. Furthermore Fubini's theorem shows that $\mathscr{S}$ is ρ-transient iff $\int_0^\infty x^{-1}\phi(\mathrm{d}x) < \infty$ and is ρ-positive if $\phi(\{0\}) > 0$, but ρ-null if $\phi(\{0\}) = 0$ and $\int_0^\infty x^{-1}\phi(\mathrm{d}x) = \infty$. These results were obtained for the case $\rho = 0$ in [13]. In this case it was also shown that $\{\pi_j\}$ is always an invariant measure. The following proposition extends this result.

Proposition 5. *The sequence* $\{\pi_j Q_j(\rho)\}$ *is a* ρ*-invariant measure and* $\{Q_j(\rho)\}$ *is a* ρ*-invariant function for* $\{X(t)\}$.

proof. From the theory of orthogonal polynomials it is known that $Q_j(\rho)>0$ and consequently we can define the polynomials $R_j(x)=Q_j(x+\rho)/Q_j(\rho)$. Let $\zeta_j=\lambda_j Q_{j+1}(\rho)/Q_j(\rho)$ $(j\in\mathscr{S})$ and $w_j=\mu_j Q_{j-1}(\rho)/Q_j(\rho)$ $(j\in\mathscr{T})$. These define a generator $\boldsymbol{R}$. It is easily seen that $\{R_j(x)\}$ is generated, according to the prescription (29), by the sequences $\{\zeta_j\}$ and $\{w_j\}$. Furthermore (30) shows that $\{R_j(x)\}$ is orthogonal with respect to ϕ. The construction can be reversed, that is, if $\tilde{\phi}$ is another measure such that $\{R_j(x)\}$ is orthogonal with respect to $\tilde{\phi}$ then $\{Q_j(x)\}$ is orthogonal with respect to $\tilde{\psi}$ where $\tilde{\psi}([0,\rho))=0$ and $\tilde{\psi}(E)=\tilde{\phi}(E-\rho)$ if $E\subset[s,\infty]$. However (27) implies that ψ is the unique solution of the $\boldsymbol{Q}$-moment problem and hence ϕ is the unique solution of the $\boldsymbol{R}$-moment problem. If follows that if $M_j=\pi_j Q_j^2(\rho)$ then

$$\tilde{p}_{ij}(t)=M_j\int_0^\infty e^{-xt}R_i(x)R_j(x)\phi(\mathrm{d}x) \tag{31}$$

defines a substochastic transition semi-group and in view of Theorem 16 of [12] this semi-group is stochastic iff

$$\sum_{j=0}^{\infty}(\xi_j M_j)^{-1}\sum_{i=0}^{j}M_i=\infty.$$

Suppose this is not the case. It then follows from Lemma 6 of [12] that $\lim_{j\to\infty}R_j(x)<\infty$ $(x<0)$. Choose $x=-\rho$. Then it follows that $\lim_{j\to\infty}Q_j(\rho)>0$ and hence that (28) is violated, a contradiction.

From the definitions above

$$\tilde{p}_{ij}(t)=(Q_j(\rho)/Q_i(\rho))\,e^{\rho t}p_{ij}(t)$$

and the condition $\sum_{j\geq 0}\tilde{p}_{ij}(t)=1$ implies that $\{Q_j(\rho)\}$ is a ρ-invariant function of $\{X(t)\}$. Furthermore $\{M_j\}$ is an invariant measure for $[\tilde{p}_{ij}(t)]$ and hence $\{\pi_j Q_j(\rho)\}$ is a ρ-invariant measure for $[p_{ij}(t)]$.

The following discussion contains the proof of

Theorem 7. *The CAD always exists for the birth and death process* $\{X(t)\}$. *It is non-defective iff* $\{X(t)\}$ *is* ρ*-recurrent and is given by* (35), (33) *and* (34) *below.*

The proof of Proposition 5 shows that $Q_j(\rho)\downarrow 0$ $(j\to\infty)$. When $\{X(t)\}$ is ρ-transient the constructions in Section 2 show that $\{m_j\}$ and $\{x_j\}$ are strictly ρ-subinvariant. Thus a ρ-transient birth and death process has (at least) two distinct ρ-subinvariant measures (functions); see [17, end of Section 3]. It also follows from (3) that when $\mathscr{S}$ is ρ-recurrent,

$$\hat{g}_{0j}(-\rho)=\lambda_0 G_{1j}(\rho)=\pi_j Q_j(\rho). \tag{32}$$

In the ρ-transient case

$$\lambda_0 G_{1j}(\rho) = \hat{p}_{0j}(-\rho)/\hat{p}_{00}(-\rho)$$

$$= \pi_j Q_j(\rho) \frac{\int_0^\infty R_j(x) x^{-1} \phi(\mathrm{d}x)}{\int_0^\infty x^{-1} \phi(\mathrm{d}x)}$$

and the integral in the numerator is

$$S_j = \sum_{k=j}^{\infty} [\lambda_k \pi_k Q_k(\rho) Q_{k+1}(\rho)]^{-1}; \tag{33}$$

see [13, Section 9.A]. The series is finite iff S is ρ-transient. Thus we have

$$G_{1j}(\rho) = \lambda_0^{-1} \pi_j Q_j(\rho)(1 - \sigma_j/S_0) \tag{34}$$

where

$$\sigma_j = \sum_{k=0}^{j-1} [\lambda_k \pi_k Q_k(\rho) Q_{k+1}(\rho)]^{-1}.$$

This expression is valid in all cases.

It is shown in [13] that

$$p_{ij}(t)/p_{kl}(t) \to Q_i(\rho)Q_j(\rho)\pi_j / Q_k(\rho)Q_1(\rho)\pi_1 \quad (t \to \infty).$$

For $s > 0$,

$$p_{0j}(t+\tau)/p_{00}(t) = \mathrm{e}^{-\rho\tau}(1 - H)$$

where

$$H = \int_\rho^\infty \mathrm{e}^{-t(x-\rho)}(1 - \mathrm{e}^{-\tau(x-\rho)})\psi(\mathrm{d}x) \Big/ \int_\rho^\infty \mathrm{e}^{-t(x-\rho)}\psi(dx)$$

and it follows as in [13, p. 389] that $H \to 0$ $(t \to \infty)$. Thus the birth and death process $\{X(t)\}$ *always* has the SRLP.

The manipulations in Section 3 carry through here, but with some changes in the ρ-transient case and we find that the CAD always exists and

$$a_j(t) = \left((G_{1j}(\rho))^{-1} \int_0^t \mathrm{e}^{\rho s} r_{1j}(s)\,\mathrm{d}s\right)(1 - \sigma_j/S_0). \tag{35}$$

In particular we see that in the ρ-transient case the CAD is defective and that $P(A(j) = \infty) \uparrow 1$ $(j \to \infty)$.

The reversibility of $\{X(t)\}$, that is, $\pi_i p_{ij}(t) = p_{ji}(t)\pi_j$, implies that when the process is recurrent, $A(j)$ has the same distribution as that of the first passage time from $\{j\}$ to $\{0\}$. This fact is exploited in the context of diffusion processes in [31]. Finally we mention that in the ρ-recurrent case the proof of Proposition 5 shows that

$$E(e^{-\theta A(j)}) = \left(\int_0^\infty e^{-\theta t}\tilde{p}_{0j}(t)\,dt\right) \Big/ M_j \int_0^\infty e^{-\theta t}\tilde{p}_{00}(t)\,dt,$$

that is, we can work in terms of a recurrent process which has the same CAD. The preceding remark shows that in principle it suffices to consider only the recurrent case in seeking limit theorems for $A(j)$ for large j. In particular it is necessary to be able to find the asymptotic behavior of $\{Q_j(\rho)\}$ in order to find that of $\{\xi_j\}$ and $\{w_j\}$.

6. Some special birth–death processes

We shall now obtain a variety of limit theorems for $A(j)$ with attention being confined mainly to very specific examples which are of common occurrence. We shall also see that sometimes it is more convenient to work with the return process and at other times with the absorbing process. In the sequel the symbols $J_\nu(\cdot)$, $I_\nu(\cdot)$ and $K_\nu(\cdot)$ will always stand for Bessel functions.

6.1. M/M/1 queue

Here $\lambda_j = \lambda$ $(j \in \mathscr{S})$ and $\mu_j = \mu$ $(j \in \mathscr{T})$. Let $r = \lambda/\mu$. It is well known that the return process, that is, the queue length process, is positive recurrent when $r < 1$, null when $r = 1$ and transient when $r > 1$. In fact when $r > 1$ the spectral measure has a density which is supported on a finite interval bounded away from the origin [14] and hence $\mathscr{S}$ is ρ-transient. We shall consider only the case $r \leq 1$. Clearly $G_{1j} = \lambda^{-1} r^j$ and it is known that

$$r_{1j}(t) = r^{(j-1)/2} e^{-(\lambda+\mu)t}\{I_{j-1}(2t(\lambda_\mu)^{1/2}) - I_{j+1}(2t(\lambda\mu)^{1/2})\};$$

see [10]. Using the transform relation

$$\int_0^\infty e^{-\theta x} I_j(ax)\,dx = a^{-j}(\theta - (\theta^2 - a^2)^{1/2})^j (\theta^2 - a^2)^{-1/2}$$

it is an easy matter to prove

Theorem 8. *If $r < 1$, then*

$$P(A(j) \leq jx) \to 1 - \exp(-x/(\mu - \lambda)) \quad (x > 0)$$

and if $r = 1$, then

$$P(A(j) \leq j^2 x) \to F_{1/2}(\lambda x) \quad (x > 0)$$

where $F_{1/2}(x)$ is the stable DF of index $\frac{1}{2}$ *whose density is*

$$f_{1/2}(x) = (2(\pi x^3)^{1/2})^{-1} \exp\left(-\frac{1}{4}x\right) \quad (x > 0).$$

6.2. $M/M/\infty$ queue or immigration–death process

Here $\lambda_j = \delta$ and $\mu_j = \mu j$ ($j \in \mathcal{S}$). The return process is always positive recurrent and

$$\begin{aligned}\phi(s,t) &= \sum_{j=0}^{\infty} p_{0j}(t) s^j \\ &= \exp[-c(1-e^{-\mu t})(1-s)]\end{aligned} \tag{36}$$

where $c = \delta/\mu$ and $G_{1j} = c^j/\lambda j!$.

By taking the Laplace transform of (36) we obtain

$$\hat{p}_{0j}(\theta) = (\mu j!)^{-1} \int_0^1 y^{\theta/\mu - 1}\, e^{-c(1-y)} [c(1-y)]^j \,dy$$

and hence

$$\alpha_j(\theta) = \frac{j^{-\theta/\mu} \int_0^j x^{\theta/\mu-1}\, e^{cx/j} (1-x/j)^j \,dx}{\int_0^1 y^{\theta/\mu-1}\, e^{cy}\,dy}$$

whence:

Theorem 9. *$P(A(j) - \mu^{-1}\log j \le x) \to F(x)$ $(-\infty < x < \infty)$ where*

$$\begin{aligned}\phi(\theta) &= \int_{-\infty}^{\infty} e^{-\theta x}\, dF(x) \\ &= \Gamma(\theta/\mu) \Big/ \int_0^1 y^{\theta/\mu-1}\, e^{cy}\,dy \quad (\theta > 0).\end{aligned}$$

Integration by parts yields

$$\phi(\theta) = \frac{\Gamma(1+\theta/\mu)}{e^c - c\int_0^1 y^{\theta/\mu}\, e^{cy}\,dy} \quad (\theta > -\mu^{-1}) \tag{37}$$

and it is now clear that $\phi(\theta)$ is a non-defective moment generating function; the requisite continuity theorem can be found in [23]. Now

$$\Gamma(1+\theta/\mu) = \int_{-\infty}^{\infty} e^{-\theta x}\, dV(x)$$

where $V(x) = \exp(-e^{-\mu x})$ is an extreme value DF. Moreover the derivative of the denominator in (37) is completely monotone and hence it follows [9, p. 441] that the reciprocal of this denominator is the Laplace-Stieltjes transform of a DF, $W(x)$ say, and $W(0+) = 0$. This gives the representation

$$F(x) = \int_0^{\infty} V(x-y)\, dW(y).$$

6.3. Null recurrent linear birth–death-immigration process

Here $\lambda_j = \lambda j + \delta$, $\mu_j = \mu j$ ($j \in \mathcal{S}$). The theory of Markov branching processes with immigration [22] shows that $\mathcal{S}$ is positive recurrent when $r = \lambda/\mu < 1$, null if $r = 1$ and $\delta \leqslant \lambda$, transient if $r = 1$ and $\delta > \lambda$ and $\delta(1 - r^{-1})$-positive if $r > 1$. We shall confine our attention to the null case.

It is easily shown that

$$\begin{aligned}\phi(s, t) &= \sum_{j=0}^{\infty} p_{0j}(t) s^j \\ &= (1 + \lambda t(1-s))^{-\delta/\lambda}.\end{aligned}$$

We first consider the boundary case $\delta = \lambda$ and prove Theorem 10 below. Under this condition we obtain [1, p. 1029]

$$\hat{p}_{00}(\theta) = \lambda^{-1} \mathrm{e}^{\theta/\lambda} E_1(\theta/\lambda)$$

where $E_1(\cdot)$ is the exponential integral [1, p. 228] whence [1, p. 229]

$$\hat{p}_{00}(\theta) \sim \lambda^{-1} \log \theta^{-1} \quad (\theta \downarrow 0). \tag{38}$$

The spectral measure has density $\lambda^{-1} \mathrm{e}^{-x/\lambda}$ ($x \geqslant 0$) and $Q_j(x) = L_j(x/\lambda)$ where $L_j(\cdot)$ is a Laguerre polynomial; see [15, p. 652]. Making use of the integral representation [1, p. 785]

$$L_j(x) = (j!)^{-1} \mathrm{e}^{x} \int_0^{\infty} \mathrm{e}^{-t} t^j J_0(2(tx)^{1/2})\, \mathrm{d}t$$

we obtain

$$\begin{aligned}\hat{p}_{0j}(\theta) &= (\pi_j/\lambda j!) \int_0^{\infty} \mathrm{e}^{-u} u^j \left(\int_0^{\infty} \frac{J_0(2(ux/\lambda)^{1/2})}{\theta + x}\, \mathrm{d}x \right) \mathrm{d}y \\ &= \frac{2\pi_j}{\lambda j!} \int_0^{\infty} \mathrm{e}^{-u} u^j \left(\int_0^{\infty} \frac{y J_0(y)}{y^2 + 4\theta u/\lambda}\, \mathrm{d}y \right) \mathrm{d}u \\ &= \frac{2\pi_j}{\lambda j!} \int_0^{\infty} \mathrm{e}^{-u} u^j K_0(2(\theta u/\lambda)^{1/2})\, \mathrm{d}u.\end{aligned}$$

Substitute θ/n_j for θ where $0 < n_j/j \to \infty$ and denote the resulting coefficient of π_j by C_j. Choose $\varepsilon > 0$ and break the range of integration into the sets $[0, \varepsilon n_j]$ and $[\varepsilon n_j, \infty)$ and write $C_j = C_{1j} + C_{2j}$. Since $K_0(\cdot)$ is non-increasing

$$\begin{aligned}C_{2j} &\leqslant (2/\lambda j!) K_0(2(\theta\varepsilon/\lambda)^{1/2}) \int_{\varepsilon n_j}^{\infty} \mathrm{e}^{-u} u^j\, \mathrm{d}u \\ &= \mathrm{O}\left[(j^{j+1}/j!) \int_{\varepsilon n_j/j}^{\infty} \mathrm{e}^{-j(v-1)} v^j\, \mathrm{d}v \right].\end{aligned}$$

We shall choose $n_j = j^{1+v}$, $v > 0$ and it is not difficult to see that $\mathrm{O}[\cdot] \to 0$ ($j \to \infty$) by observing that $\mathrm{e}^{-j(v-1)} = \mathrm{O}((j+k)! j^{-j-k} v^{-j-k})$ and choosing $kv > 1$.

By choosing ε small enough we can replace $K_0(2(\theta u/\lambda n_j)^{1/2})$ in C_{1j} by $\log(\frac{1}{2}(\lambda n_j/\theta u)^{1/2})$ with arbitrarily small relative error. This yields

$$C_{1j} \sim ((\log n_j)/\lambda j!) \int_0^{\varepsilon n_j} e^{-u} u^j \, du - (\lambda j!)^{-1} \int_0^{\varepsilon n_j} (\log u) u^j e^{-u} \, du. \tag{39}$$

The first term $\sim\lambda^{-1}\log n_j$ and the argument used above yields

$$j! \int_{\varepsilon n_j}^{\infty} (\log u) u^j e^{-u} \, du = O(\log j) \quad (j \to \infty).$$

Now

$$\int_0^{\infty} u^j e^{-u} \log u \, du = \Gamma'(j+1) \sim j! \log j \tag{40}$$

[1, pp. 257–259] and using (38)–(40) we find that

$$\alpha_j(\theta j^{-1-v}) \to v/(1+v) \quad (v > 0).$$

The limit is the Laplace–Stieltjes transform of a distribution concentrating mass $v/(1+v)$ at the origin and no mass in any finite interval bounded away from the origin. Thus

$$P(A(j) j^{-1-v} \leqslant 1) \to v/(1+v)$$

and the next result now follows.

Theorem 10. *For the linear birth–death–immigration process with $\lambda = \mu = \delta$,*

$$P(\log A(j) \leqslant jx) \to (1 - x^{-1})^+ \quad (j \to \infty).$$

In dealing with the case $\delta < \lambda$ we can expand our scope. Consider a general birth-death process for which

$$\pi_j \sim D j^{\gamma-1}, \quad (\lambda_j \pi_j)^{-1} \sim C j^{\beta-1},$$

where $C, D, \beta, \gamma > 0$. Let $\alpha = \beta/(\beta+\gamma)$. It is shown in [16] that

$$Q_j(x j^{-\beta-\gamma}) \to \Gamma(1-\alpha) \left(\frac{(CDx)^{1/2}}{\beta+\gamma} \right)^{\alpha} J_{-\alpha}\left(\frac{2(CDx)^{1/2}}{\beta+\gamma} \right) = N(x), \text{ say}$$

pointwise for $x \geqslant 0$ and uniformly in compact subsets of $[0, \infty)$. It is also shown that

$$\hat{p}_{00}(\theta) \sim H\theta^{-\alpha} \quad (\theta \downarrow 0),$$

where H is a positive constant, and a Tauberian theorem for Stieltjes transforms (e.g., the dual version of Theorem 2 in [28, p. 61]) shows that

$$\psi(x) \sim (H/\Gamma(\alpha)\Gamma(2-\alpha)) x^{1-\alpha} \quad (x \downarrow 0).$$

Thus for $\theta > 0$ the measure $\psi(\theta j^{-\beta-\gamma}\,\mathrm{d}x)/\psi(\theta j^{-\beta-\gamma})$ converges $(j \to \infty)$ to a measure whose density is $(1-\alpha)x^{-\alpha}$. Since

$$\alpha_j(\theta j^{-\beta-\gamma}) = (j^{\beta+\gamma}/\theta\hat{p}_{00}(\theta j^{-\beta-\gamma}))\int_0^{\infty} \frac{Q_j(x\theta j^{-\beta-\gamma})}{1+x}\psi(\theta j^{-\beta-\gamma}\,\mathrm{d}x)$$

it eventually follows that

$$\alpha_j(\theta j^{-\beta-\gamma}) \to \alpha(\theta) = (\Gamma(\alpha)\Gamma(1-\alpha))^{-1}\int_0^{\infty} \frac{N(\theta x)}{1+x}x^{-\alpha}\,\mathrm{d}x.$$

Let $B = 2(CD)^{1/2}/(\beta+\gamma)$ and use the transformation $y = B(\theta x)^{1/2}$ to give an integral which is tabulated [1, 11.4.44]. We obtain

$$\alpha(\theta) = (2/\Gamma(\alpha))(\tfrac{1}{2}B\theta^{1/2})^{\alpha}K_{-\alpha}(B\theta^{1/2}).$$

The integral representation

$$K_{-\alpha}(x) = \frac{\pi^{1/2}(x/2)^{-\alpha}}{\Gamma(1/2-\alpha)}\int_1^{\infty} \mathrm{e}^{-xt}(t^2-1)^{-\alpha-1/2}\,\mathrm{d}t$$

holds if $\alpha < \frac{1}{2}$. This leads to a Laplace transform representation for $\alpha(\theta)$ and analytic continuation extends this to $\alpha < 1$ whence

Theorem 11. *$P(j^{-\beta-\gamma}A(j) \leq x) \to G_\alpha(x)$ $(j \to \infty;\ x \geq 0)$ where $G_\alpha(\cdot)$ has the density*

$$g_\alpha(x) = \left(\frac{B}{2}\right)^{2\alpha}\frac{x^{-1-\alpha}}{\Gamma(\alpha)}\mathrm{e}^{-B^2/4x} \quad (z > 0).$$

It is easy to see that $B = 2\lambda^{-1/2}$ and $\alpha = 1 - \delta/\gamma$ for the linear birth-death-immigration process.

6.4. Linear birth–death process

We let $\{Z(t)\}$ be the linear birth–death process with positive birth and death rates, λ and μ, respectively. Referring to the notation of Section 4, $\nu = \lambda + \mu$, $m = 2\lambda/\nu$, $\Delta = \lambda - \mu$ and when $m > 1$, $q = \mu/\lambda$ and $\delta = \Delta$. In this case $w(t) = (1-\mu/\lambda)\,\mathrm{e}^{-t}$ and the density of the limit distribution of Theorem 5 is

$$(\Gamma(1-\rho/\Delta))^{-1}\,\mathrm{e}^{-x(1-\rho/\Delta)}\exp(-\mathrm{e}^{-x}).$$

When $m < 1$ the dual "$\tilde{\ }$" process exists and is the linear birth and death process with birth rate μ and death rate λ. It follows that the limit distribution of Theorem 6 is the extreme value distribution whose DF is $\exp(-\mathrm{e}^{-x})$. The occurrence of extreme value distributions when $\lambda \leq \mu$ is not surprising in view of the fact that $\{Z(t)\}$ is a reversible process and, as pointed out in Section 5, the CAD is the same as the extinction time distribution when $Z(0) = j$. However it follows from the independence of family lines that this latter distribution is that of the maximum of j independent copies of T.

When $\lambda < \mu$ we can write

$$F_1(s,t) = \frac{(1-s)-(1-rs)(\exp(-\Delta t))}{r(1-s)-(1-rs)(\exp(-\Delta t))},$$

where $r = \lambda/\mu$, and

$$\int_0^t (F_1(s,\tau) - F_1(0,\tau))\,\mathrm{d}\tau = ((1-r)/\Delta r)\log[1 - rs(1-\mathrm{e}^{\Delta t})/(1 - r\,\mathrm{e}^{\Delta t})].$$

Thus $\dot{G}_{1j} = (\lambda j)^{-1} r^j$ and

$$a_j(t) = [(1-\mathrm{e}^{\Delta t})/(1 - r\,\mathrm{e}^{\Delta t})]^j.$$

This yields

$$\begin{aligned}\mu_j^{(1)} &= \int_0^\infty (1 - a_j(t))\,\mathrm{d}t \\ &= -((1-r)/\Delta) \sum_{k=0}^{j} \sum_{n=0}^{\infty} r^n (k+1+n)^{-1} \\ &\sim -\Delta^{-1} \log j.\end{aligned}$$

When $\lambda = \mu$, Levikson [20] has shown that

$$a_j(t) = j \int_0^t \frac{(\lambda y)^j}{(1+\lambda y)^{j+1}} \frac{\mathrm{d}y}{y}.$$

When $\lambda > \mu$ evaluation of the relevant integrals involves hypergeometric functions and is not very revealing. The equation for ρ can be written as

$$q_0 + \rho = q_0 \int_0^\infty \mathrm{e}^{-\rho t} r'_{10}(t)\,\mathrm{d}t$$

which in principle can be numerically solved. Finally when $\lambda < \mu$ it is known that the limiting distribution of the return process is a logarithmic series distribution [32].

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